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Fractional variational calculus and the transversality conditions

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Abstract

This paper presents the Euler–Lagrange equations and the transversality conditions for fractional variational problems. The fractional derivatives are defined in the sense of Riemann–Liouville and Caputo. The connection between the transversality conditions and the natural boundary conditions necessary to solve a fractional differential equation is examined. It is demonstrated that fractional boundary conditions may be necessary even when the problem is defined in terms of the Caputo derivative. Furthermore, both fractional derivatives (the Riemann–Liouville and the Caputo) arise in the formulations, even when the fractional variational problem is defined in terms of one fractional derivative only. Examples are presented to demonstrate the applications of the formulations.

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1. Introduction

Integer variational calculus plays a significant role in many areas of science, engineering and applied mathematics [1, 2]. In many applications, it is used to obtain the laws governing the physics of systems and boundary/terminal conditions [3, 4]. It has been the starting point for various numerical schemes such as Ritz, finite difference and finite element methods [2, 5]. In optimal control, it is used to obtain the differential equations and the terminal conditions for optimal trajectory of a system [6, 7].

Although significant work has been done in the area of integer variational calculus, very little has been done in the area of fractional variational calculus. Recently Riewe [8, 9] developed Lagrangian, Hamiltonian and other concepts of classical mechanics for nonconservative systems. Agrawal [10] presented a heuristic approach to obtain differential equations of fractionally damped systems. Later, Agrawal [11] presented generalized Euler–Lagrange equations for unconstrained and constrained fractional variational problems. Klimek presented a fractional sequential mechanics model with symmetric fractional derivatives [12] and stationary conservation laws for fractional differential equations with variable coefficients [13]. Dreisigmeyer and Young [14] presented nonconservative Lagrangian mechanics using a generalized function approach. In [15] the authors show that obtaining differential equations for a nonconservative system using fractional variational calculus may not be possible.

The fractional Euler–Lagrange equation has recently been used by Baleanu and co-workers to model fractional Lagrangian and Hamiltonian formulations with linear velocities [16, 17] and Hamiltonian equations for fractional variational problems [18]. Agrawal [19, 20] presented formulations for deterministic and stochastic analyses of fractional optimal control problems. Tarasov and Zaslavsky [21] have used variational Euler–Lagrange equations to derive fractional generalization of the Ginzburg–Landau equation for fractal media. Stanislavsky [23] presented a generalized formulation for fractional systems.

It should be noted that the integer variational calculus provides not only the Euler–Lagrange equations but also the transversality conditions, which lead to natural boundary conditions for the minimization of the associated functional [2, 3]. In contrast, the current state of research in the field of fractional variational calculus is largely focused on obtaining the Euler–Lagrange equations. Like in classical Lagrangian and Hamiltonian mechanics, the necessary boundary conditions are taken from physical considerations. These conditions may not lead to extremum of the functional. To the author's knowledge, no attention has been given to transversality and natural boundary conditions for fractional variational problems. For boundary value problems, when sufficient kinematic boundary conditions are not specified, the natural boundary conditions are necessary to solve a problem analytically. For fractional variational problems, the transversality conditions and the natural boundary conditions are not obvious.

The transversality conditions for fractional variational problems are expected to provide additional information. It is well known that the solutions of the fractional differential equations defined in terms of Riemann–Liouville require fractional initial conditions [22]. It is believed by many that fractional initial conditions are not physical. Therefore, to overcome this problem, the initial conditions are generally taken as zero. This belief may be largely due to the fact that fractional derivatives have not been fully assimilated in science and engineering. For example, note that under certain conditions, $\partial^{1/2}T/\partial t^{1/2}$ represents heat flux at a boundary of a 1D heat conduction problem. Here T and t represent temperature and time, respectively. Heat flux is a well-accepted concept. Thus, fractional boundary conditions may be a reality. Sometimes in engineering and physics, the phrase *generalized force* is used in an extended sense of the definition to mean both force and moment. As discussed later, the transversality conditions for fractional variational problem will allow us to extend the definition of many of the physical terms even further.

It should be noted that the fractional differential equations defined in terms of the Caputo derivatives require the regular boundary conditions. Therefore, one may presume that a fractional variational problem defined in terms of the Caputo derivatives may not require fractional initial conditions. As discussed below, this safe haven may not be so safe. The point to be made here is that the fractional terminal conditions (like initial conditions for a dynamics problem and the boundary conditions in mechanics) may be necessary to solve a fractional calculus problem which has not been adopted in formulations and modelling of systems so far. The transversality and natural boundary conditions derived here motivate us to include the fractional boundary conditions in the formulations.

In this paper we develop the Euler–Lagrange equations and the transversality conditions for fractional variational problems defined in terms of Riemann–Liouville and Caputo derivatives. These transversality conditions suggest the appropriate boundary conditions to solve a fractional variational problem. It is demonstrated that fractional boundary conditions may be necessary even when the problem is defined in terms of Caputo derivatives only. The relationship between the boundary conditions arising from the transversality conditions and those required by the Laplace transform technique is examined. Further, both fractional derivatives (the Riemann–Liouville and the Caputo) may arise in the formulation, even when the fractional variational problem is defined in terms of only one type of fractional derivative.

2. Fractional derivatives and their Laplace transforms

Several definitions have been proposed for a fractional derivative. We will deal with the Riemann–Liouville and the Caputo fractional derivatives only. In this section, we present the definitions of these two derivatives and their Laplace transforms. We also discuss the types of boundary conditions that are necessary to solve a fractional differential equation. Most of the equations presented in this section could be found with some minor notational changes in [22, 24]. They are presented here for completeness and for ease in the discussion to follow.

We begin with the left Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y(x)$ which is defined as [24]

$$ {}_0I_x^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} y(\tau)\,\mathrm{d}\tau, \qquad x, \alpha > 0, \tag{1} $$

where $\Gamma(*)$ represents the Gamma function. This integral can be written as the Laplace convolution between $y(x)$ and $\Phi_\alpha(x)$ as

$$ {}_0I_x^\alpha y(x) = \Phi_\alpha(x) * y(x) = \int_0^x \Phi_\alpha(x-\tau) y(\tau)\,\mathrm{d}\tau, \tag{2} $$

where * is the convolution operator and the function $\Phi_\alpha(x)$ is defined as

$$ \Phi_\alpha(x) = \frac{1}{\Gamma(\alpha)} \begin{cases} 0, & x \leqslant 0 \\ x^{\alpha-1}, & x > 0. \end{cases} \tag{3} $$

Using (1) the left Riemann–Liouville derivative ${}_0D_x^\alpha y(x)$ and the left Caputo derivative ${}_0^CD_x^\alpha y(x)$ of order $\alpha > 0$ are given as

$$ {}_0D_x^\alpha y(x) = D^n\, {}_0I_x^{n-\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_0^x (x-\tau)^{n-\alpha-1} y(\tau)\,\mathrm{d}\tau, \quad n-1 < \alpha < n, \tag{4} $$

and

$$ {}_0^CD_x^\alpha y(x) = {}_0I_x^{n-\alpha} D^n y(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n y(\tau)\,\mathrm{d}\tau, \quad n-1 < \alpha < n, \tag{5} $$

where $D = \mathrm{d}/\mathrm{d}x$ represents the ordinary derivative and n is an integer. When α is an integer, these derivatives represent the ordinary derivatives. The right fractional integral and the right Riemann–Liouville and the right Caputo derivatives will be defined in the next section. It should be pointed out that in the literature, the fractional integral, the Riemann–Liouville and the Caputo derivatives generally mean the left fractional integral, the left Riemann–Liouville and the left Caputo derivatives, respectively.

Using (1) to (5) and the properties of the convolution integral, the Laplace transforms of ${}_0D_x^\alpha y(x)$ and ${}_0^\alpha D_x^\alpha y(x)$ are given as [24]

$$ L\left[{}_0D_x^\alpha y(x)\right] = s^\alpha Y(s) - \sum_{k=0}^{n-1} D^k\, {}_0I_x^{n-\alpha} y(0^+) s^{n-1-k} \tag{6} $$

and

$$L\left[{}_0^C D_x^\alpha y(x)\right] = s^\alpha Y(s) - \sum_{k=0}^{n-1} D^k y(0^+) s^{\alpha-1-k}, \tag{7}$$

where s is the Laplace parameter and $Y(s) = L[y(x)]$ is the Laplace transform of $y(x)$. Note that the Laplace transform of ${}_0D_x^\alpha y(x)$ contains the fractional initial conditions whereas the Laplace transform of ${}_0^C D_x^\alpha y(x)$ contains the regular initial conditions. Therefore, (6) suggests that the solution of a linear fractional differential equation defined in terms of the left Riemann–Liouville derivatives will require fractional initial conditions. On the other hand, (7) suggests that the solution of a linear fractional differential equation defined in terms of the left Caputo derivatives will require regular initial conditions. It will be shown that a fractional variational problem may require fractional initial conditions even when the problem is defined in terms of the left Caputo derivatives.

3. The generalized Euler–Lagrange equations and the transversality conditions

In this section, we present the generalized Euler–Lagrange equations and the transversality conditions for fractional variational problems defined in terms of the Riemann–Liouville and the Caputo derivatives. We begin with the right Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y(x)$ which is defined as [24]

$${}_xI_1^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_x^1 (\tau - x)^{\alpha-1} y(\tau)\, \mathrm{d}\tau, \qquad x, \alpha > 0. \tag{8}$$

For simplicity, we have taken the upper limit of the integral as 1. However, the upper limit can be any value greater than x and less than infinity.

Using (8), the right Riemann–Liouville and the right Caputo fractional derivatives are given, respectively, as

$${}_xD_1^\alpha y(x) = (-D)^n {}_xI_1^{n-\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_x^1 (\tau - x)^{n-\alpha-1} y(\tau)\, \mathrm{d}\tau, \quad n-1 < \alpha < n \tag{9}$$

and

$${}_x^C D_1^\alpha y(x) = {}_xI_1^{n-\alpha} (-D)^n y(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^1 (\tau - x)^{n-\alpha-1} \left(-\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n y(\tau)\, \mathrm{d}\tau, \quad n-1 < \alpha < n. \tag{10}$$

When α is an integer, these derivatives are replaced with $(-D)^\alpha$. It will be seen shortly that these derivatives arise when the functionals to be minimized are defined in terms of the left Riemann–Liouville and the left Caputo fractional derivatives.

We now consider the following fractional variational problem containing the left Riemann–Liouville fractional derivative only. Among all possible functions $y(x)$, find the function $y^*(x)$, which minimizes the functional

$$J[y] = \int_0^1 F\left(x, y, {}_0D_1^\alpha y\right) \mathrm{d}x \tag{11}$$

and satisfies the condition

$$y(0) = y_0. \tag{12}$$

This problem is the same as that considered in [11] with two exceptions. First, it does not include the right Riemann–Liouville fractional derivative. This choice is made for simplicity.

Second, in this problem, the boundary condition is specified only at $x = 0$ so that we can develop the natural boundary condition. For simplicity, we also assume that $0 < \alpha < 1$ and that all differentiability conditions are met. We further assume that the end points are specified. Here, the function value is given at one end ($x = 0$) but free at the other end ($x = 1$). The case where an end point lies on a prescribed curve will be considered in the future.

Using the approach presented in [11], it can be demonstrated that for $J[y]$ to have an extremum, the following conditions must be satisfied,

$$\int_0^1 \left[\frac{\partial F}{\partial y} + {}_x^C D_1^\alpha \frac{\partial F}{\partial\, {}_0D_x^\alpha y}\right] \delta y \, \mathrm{d}x + \left(\frac{\partial F}{\partial\, {}_0D_x^\alpha y}\right) \delta\, {}_0D_x^{\alpha-1} y(x)\bigg|_0^1 = 0, \tag{13}$$

where $\delta(*)$ is the variation operator and ${}_0D_x^{\alpha-1} y(x)$ must be interpreted as the fractional integral of order $1 - \alpha$. Since δy is arbitrary, it follows from a well-established result in calculus of variations that [2]

$$\frac{\partial F}{\partial y} + {}_x^C D_1^\alpha \frac{\partial F}{\partial\, {}_0D_x^\alpha y} = 0, \tag{14}$$

and

$$\left(\frac{\partial F}{\partial\, {}_0D_x^\alpha y}\right) \delta\, {}_0D_x^{\alpha-1} y(x) = 0, \qquad x = 0, 1. \tag{15}$$

Equations (14) and (15) are the generalized Euler–Lagrange equation [11] and the transversality conditions for the fractional variational problem defined in terms of the left Riemann–Liouville fractional derivative. Equation (15) suggests that either

$$\left(\frac{\partial F}{\partial\, {}_0D_x^\alpha y}\right) = 0, \qquad x = 0, 1 \tag{16}$$

or

$$\delta\, {}_0D_x^{\alpha-1} y(x) = 0, \qquad x = 0, 1, \tag{17}$$

i.e. ${}_0D_x^{\alpha-1} y(x)$ at the end points should be specified. These boundary conditions are fractional and they are similar to those required when the Laplace transform technique is used. Since y at $x = 1$ is not specified, it follows that

$$\left(\frac{\partial F}{\partial\, {}_0D_x^\alpha y}\right)\bigg|_{x=1} = 0. \tag{18}$$

Equation (18) is called the natural boundary conditions, and to obtain the optimum solution, this condition must be satisfied. In many applications, the natural boundary conditions may have some physical interpretations. One may then ask for a physical interpretation of (18). However, this will depend on the physics of the problem. In the examples considered, extensions of some physical definitions for the natural boundary conditions are given.

Note that (14) is somewhat different from that presented in [11]. It contains a Caputo fractional derivative even when the functional in (11) contains no such term. This is because some of the boundary conditions are not specified. Equation (11) can be written purely in terms of the Riemann–Liouville fractional derivative. However, in that case, the resulting equations will contain some extra terms.

We now consider the following fractional variational problem containing the left Caputo fractional derivative. Among all possible curve $y(x)$, find the curve $y^*(x)$, which minimizes the functional

$$J[y] = \int_0^1 F\left(x, y, {}_0^C D_1^\alpha y\right) \mathrm{d}x \tag{19}$$

and satisfies the initial condition given by (12). Once again, we assume that $0 < \alpha < 1$ and that all differentiability conditions are met. We also assume that the end points are fixed. The approach presented in [11] can be used with some minor changes for Caputo derivative to obtain the optimality conditions for this case also. This leads to

$$\int_0^1 \left[\frac{\partial F}{\partial y} + {}_xD_1^\alpha \frac{\partial F}{\partial\, {}_0^CD_x^\alpha y} \right] \delta y \,\mathrm{d}x + \left({}_xD_1^{\alpha-1} \frac{\partial F}{\partial\, {}_0^CD_x^\alpha y} \right) \delta y(x) \Big|_0^1 = 0. \tag{20}$$

Since δy is arbitrary, it follows from a well-established result in calculus of variations that [2]

$$\frac{\partial F}{\partial y} + {}_xD_1^\alpha \frac{\partial F}{\partial\, {}_0^CD_x^\alpha y} = 0 \tag{21}$$

and

$$\left({}_xD_1^{\alpha-1} \frac{\partial F}{\partial\, {}_0^CD_x^\alpha y} \right) \delta y(x) \Big|_0^1 = 0, \qquad x = 0, 1. \tag{22}$$

Equations (21) and (22) are the generalized Euler–Lagrange equation [11] and the transversality conditions for the fractional variational problem defined in terms of the left Caputo fractional derivative. Note that (21) contains a right Riemann–Liouville fractional derivative even when the functional does not contain any Riemann–Liouville fractional derivative term.

Equation (22) suggests that either

$$\left({}_0D_x^{\alpha-1} \frac{\partial F}{\partial\, {}_0^CD_x^\alpha y} \right) = 0, \qquad x = 0, 1 \tag{23}$$

or

$$\delta y(x)|_0^1 = 0, \qquad x = 0, 1, \tag{24}$$

i.e. $y(x)$ at the end points should be specified. The boundary conditions resulting from (24) are the kinematic boundary conditions. They have no fractional derivative terms, and thus they are consistent with those required by the Laplace transform technique. Since y at $x = 1$ is not specified, it follows that

$$\left({}_xD_1^{\alpha-1} \frac{\partial F}{\partial\, {}_0^CD_x^\alpha y} \right) \Big|_{x=1} = 0. \tag{25}$$

Equation (25) is called the natural boundary conditions and the optimum solution must satisfy this condition. Note that this condition, in general, contains fractional derivative terms. Thus, fractional variational problems defined in terms of Caputo fractional derivatives may require imposition of fractional boundary conditions.

The forgoing formulations can be extended to functionals with higher order fractional derivatives and multi-dimensional functions. In particular, if $1 < \alpha < 2$, then the transversality conditions corresponding to the fractional variational problem defined by (11) and (12) are given as

$$\left(\frac{\partial F}{\partial\, {}_0D_x^\alpha y} \right) \delta\, {}_0D_x^{\alpha-1} y(x) = 0, \qquad x = 0, 1 \tag{26}$$

and

$$\left(D \frac{\partial F}{\partial\, {}_0D_x^\alpha y} \right) \delta\, {}_0D_x^{\alpha-2} y(x) = 0, \qquad x = 0, 1 \tag{27}$$

and the generalized Euler–Lagrange equation remains the same as in (14). Equations (26) and (27) can be derived using the approach presented here and in [11]. For this reason, the derivations of these equations are not presented here. These equations suggest that either

$$\left(\frac{\partial F}{\partial\, {}_0D_x^\alpha y} \right) = 0, \qquad x = 0, 1 \tag{28}$$

or ${}_0D_x^{\alpha-1}y(x)$ at $x = 0$ and 1 must be specified and either

$$\left(D\frac{\partial F}{\partial\, {}_0D_x^{\alpha} y}\right) = 0, \qquad x = 0, 1 \tag{29}$$

or ${}_0D_x^{\alpha-2}y(x)$ at $x = 0$ and 1 must be specified. Equations (28) and (29) represent the natural boundary conditions for the fractional variational problem defined in terms of Riemann–Liouville fractional derivatives. Once again, ${}_0D_x^{\alpha-2}y(x)$ must be interpreted as a fractional integral.

Following the above approach, for $1 < \alpha < 2$, the transversality conditions corresponding to the fractional variational problem defined by (19) and (12) are given as

$$\left({}_xD_1^{\alpha-2}\frac{\partial F}{\partial\, {}_0^CD_x^{\alpha} y}\right)\delta Dy(x) = 0, \qquad x = 0, 1 \tag{30}$$

and

$$\left({}_xD_1^{\alpha-1}\frac{\partial F}{\partial\, {}_0^CD_x^{\alpha} y}\right)\delta y(x) = 0, \qquad x = 0, 1, \tag{31}$$

and the generalized Euler–Lagrange equation remains the same as in (21). Once again, we omit the derivations of (30) and (31). These equations suggest that either

$$\left({}_xD_1^{\alpha-2}\frac{\partial F}{\partial\, {}_0^CD_x^{\alpha} y}\right) = 0, \qquad x = 0, 1 \tag{32}$$

or $Dy(x)$ at $x = 0$ and 1 must be specified and either

$$\left({}_xD_1^{\alpha-1}\frac{\partial F}{\partial\, {}_0^CD_x^{\alpha} y}\right) = 0, \qquad x = 0, 1 \tag{33}$$

or $y(x)$ at $x = 0$ and 1 must be specified. Equations (32) and (33) represent the natural boundary conditions for fractional variational problem defined in terms of Caputo fractional derivatives. Note that these equations may contain fractional derivative terms.

4. Illustrative examples

In this section, we consider two examples to show some applications of the transversality conditions developed in the previous section.

4.1. Example 1

As a first example, consider the following functional:

$$J[y] = \frac{1}{2}\int_0^1 \left[ay^2 + \left({}_0D_x^{\alpha} y + y\right)^2\right] \mathrm{d}x \tag{34}$$

and the following boundary condition:

$$y(0) = 1. \tag{35}$$

We assume that $0 < \alpha < 1$. We will consider two cases.

4.1.1. Case 1. Let a be 0. In this case, the minimum value of $J[y]$ will be 0, if a function could be found that satisfies (35) and the differential equation ${}_0D_x^\alpha y + y = 0$. For this problem the Euler–Lagrange equation and the transversality condition are

$$ {}_x^C D_1^\alpha \left({}_0D_x^\alpha y + y\right) + \left({}_0D_x^\alpha y + y\right) = 0 \tag{36} $$

and

$$ \left({}_0D_x^\alpha y + y\right) = 0, \qquad \text{at} \qquad x = 1, \tag{37} $$

respectively. Applying the operator ${}_xI_1^\alpha$ on both sides of (36) and using (37), it can be demonstrated that ${}_0D_x^\alpha y + y = 0$ for $0 < x < 1$, as expected. Note that the transversality condition contains a fractional derivative term. Thus, a fractional boundary condition has been used to solve the problem.

Let us examine (36) and (37) once again. If we define $z = \left({}_0D_x^\alpha y + y\right)$, then in terms of z, the fractional boundary conditions for the problem defined by (36) and (37) completely disappear. This z could be thought of as a mapping of y in some other space. This raises a deep philosophical question: Is it that we have developed the law of physics in terms of the mapped variables (in this example z) and have not realized the more fundamental variables (in this example y)?

4.1.2. Case 2. This time, let a be 1. For this case, the Euler–Lagrange equation is

$$ {}_x^C D_1^\alpha \left({}_0D_x^\alpha y + y\right) + y + \left({}_0D_x^\alpha y + y\right) = 0 \tag{38} $$

and the transversality condition is given by (37). Solving (38) is not straightforward, and perhaps its closed form solution does not exist. This problem is equivalent to the following fractional optimal control problem [19]. Find the optimal control u that minimizes the performance index

$$ J[u] = \frac{1}{2}\int_0^1 [y^2 + u^2]\,\mathrm{d}x \tag{39} $$

and satisfies the dynamic constraint

$$ {}_0D_x^\alpha y = -y + u \tag{40} $$

and the initial condition given by (35). This problem is solved in [19] using a numerical technique. It is demonstrated that $u(1) = 0$. Using (37) and (40), it follows that this condition is consistent with the transversality condition.

4.2. Example 2

As the second example, consider the functional

$$ J[y] = \int_0^1 \left[\frac{1}{2}\left({}_0D_x^\alpha y\right)^2 - y\right] \mathrm{d}x \tag{41} $$

and the boundary condition

$$ y(0) = y_0. \tag{42} $$

We will consider two cases for this example also.

Table 1. Computed values of the natural boundary conditions.

N	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
10	5.4720×10^{-2}	2.0775×10^{-2}	1.2801×10^{-2}	7.8873×10^{-3}
20	2.3818×10^{-2}	6.8528×10^{-3}	3.6758×10^{-3}	1.9717×10^{-3}
40	1.0367×10^{-2}	2.2606×10^{-3}	1.0556×10^{-3}	4.9292×10^{-4}
80	4.5126×10^{-3}	7.4572×10^{-4}	3.0314×10^{-4}	1.2323×10^{-4}
160	1.9642×10^{-3}	2.4600×10^{-4}	8.7055×10^{-5}	3.0808×10^{-5}
320	8.5499×10^{-4}	8.1148×10^{-5}	2.5000×10^{-5}	7.7019×10^{-6}

4.2.1. Case 1. Consider that $0 < \alpha < 1$. In this case, the Euler–Lagrange equation and the natural boundary condition are

$$ {}_x^C D_1^{\alpha}\left({}_0 D_x^{\alpha} y\right) = 1 \tag{43}$$

and

$$ \left({}_0 D_x^{\alpha} y\right)\big|_{x=1} = 0, \tag{44}$$

respectively. This problem with $\alpha = 1$ and $y_0 = 0$ represents the problem of a uniformly loaded bar fixed at one end and free at the other, and in which case the transversality condition suggests that the strain at the free end should be zero. For linear materials, the stress and the strain are linearly related. Therefore, for $\alpha = 1$, (44) also suggests that stress or load at the free end should be zero. If y is the displacement, then $\mathrm{d}y/\mathrm{d}x$ is known as strain. We may call it first-order strain. Following this, ${}_0 D_x^{\alpha} y$ can be called α-order strain. For $\alpha = 1$, it will represent ordinary strain, and for $\alpha = 0$, the displacement.

In [25], a finite element technique is developed to solve the problem defined by (41) and (42) with $y_0 = 0$. The domain of $y(x)$ is discretized into several elements and the problem is solved for various values of α. Table 1 shows the computed values of the natural boundary condition for various αs and various levels of discretizations. It is clear that for each value of α, as the number of discretizations is increased, the natural boundary conditions approach 0. This is consistent with the theoretical results derived above.

4.2.2. Case 2. As a second case, consider that $1 < \alpha < 2$, and the following boundary conditions

$$ y(0) = Dy(0) = 0. \tag{45}$$

For $\alpha = 2$, this case represents a problem of a uniformly loaded cantilever beam subjected to fixed boundary at one end and free boundary at the other end.

The Euler–Lagrange equation for this case is given by (41), and the natural boundary conditions for this case are

$$ \left({}_0 D_x^{\alpha} y\right)\big|_{x=1} = D\left({}_0 D_x^{\alpha} y\right)\big|_{x=1} = 0. \tag{46}$$

For $\alpha = 2$, these conditions reduce to

$$ \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 0. \tag{47}$$

If y is the deflection of the beam, then $\mathrm{d}^2 y/\mathrm{d}x^2$ and $\mathrm{d}^3 y/\mathrm{d}x^3$ represent the bending moment and the shear force at the free end of the beam, respectively. Here, it is assumed that the material constants are 1. Thus, (47) suggests that in the case of the cantilever problem, the bending moment and the shear force at the free end must be 0. This is consistent with the classical results. In the extended sense of the definition, both $\mathrm{d}^2y/\mathrm{d}x^2$ and $\mathrm{d}^3y/\mathrm{d}x^3$ can be called the generalized forces, where $\mathrm{d}^2y/\mathrm{d}x^2$ and $\mathrm{d}^3y/\mathrm{d}x^3$ will represent generalized forces of orders 2 and 3, respectively. Following this, ${}_0D_x^\alpha y$ can be considered as the generalized force of order α.

Note that if a fractional variational problem is defined in terms of Caputo derivatives, then the natural boundary conditions may include Riemann–Liouville derivatives also. Solving such problems analytically may be difficult, so a numerical technique may be necessary. This will be considered in the future.

5. Conclusions

Generalized Euler–Lagrange equations and the transversality conditions have been presented for fractional variational problems which were defined in terms of both the Riemann–Liouville and the Caputo derivatives. It was demonstrated that both derivatives may appear in the formulation even if the problem is defined in terms of one type of derivative only. The transversality conditions gave the appropriate kinematic and natural boundary conditions. The kinematic boundary conditions obtained from the transversality conditions agreed with those required by the Laplace transform technique. It was demonstrated that the fractional boundary conditions may be necessary to solve a fractional variational problem. The natural boundary conditions may help us extend the definitions of many physical terms. Two examples were presented to demonstrate the applications of the formulations presented above.

References

[1] Bliss G A 1963 *Lectures on the Calculus of Variations* (Chicago, IL: University of Chicago Press)
[2] Gelfand I M and Fomin S V 1963 *Calculus of Variations* (Englewood Cliffs, NJ: Prentice-Hall)
[3] Dym C L and Shames 1973 *Solid Mechanics: A Variational Approach* (New York: McGraw-Hill)
[4] Goldstein H 1980 *Classical Mechanics* (Reading, MA: Addison-Wesley)
[5] Reddy J N 1984 *Energy and Variational Methods in Applied Mechanics: With an Introduction to the Finite Element Methods* (New York: Wiley)
[6] Hestenes M R 1966 *Calculus of Variations and Optimal Control Theory* (New York: Wiley)
[7] Gregory J and Lin C 1992 *Constrained Optimization in the Calculus of Variations and Optimal Control Theory* (Princeton, NJ: Van Nostrand-Reinhold)
[8] Riewe F 1996 *Phys. Rev.* E **53** 1890–9
[9] Riewe F 1997 *Phys. Rev.* E **55** 3582–92
[10] Agrawal O P 2001 *J. Appl. Mech.* **68** 339–41
[11] Agrawal O P 2002 *J. Math. Anal. Appl.* **272** 368–79
[12] Klimek M 2001 *Czech. J. Phys.* **51** 1348–54
[13] Klimek M 2001 *J. Phys A: Math. Gen.* **35** 6675–93
[14] Dreisigmeyer D W and Young P M 2003 *J. Phys. A: Math. Gen.* **36** 8297–310
[15] Dreisigmeyer D W and Young P M 2004 *J. Phys. A: Math. Gen.* **37** L117–21
[16] Baleanu D and Avkar T 2004 *Nuovo Cimento Della Societa Italiana Di Fisica B* **119** 73–9
[17] Muslih S I and Baleanu D 2005 *J. Math. Anal. Appl.* **304** 599–606
[18] Muslih S I and Baleanu D 2005 *Czech. J. Phys.* **55** 633–42
[19] Agrawal O P 2004 *Nonlinear Dyn.* **38** 323–37
[20] Agrawal O P 2005 *Fractional Differentiation and Its Applications* ed A Le Mehaute, J A T Machado, J C Trigeassou and J Sabatier pp 615–24
[21] Tarasov V E and Zaslavsky G M 2005 *Physica* A **354** 249–61
[22] Podlubny I 1999 *Fractional Differential Equations* (New York: Academic)
[23] Stanislavsky A A 2006 *Eur. Phys. J.* B **49** 93–101
[24] Gorenflo R and Mainardi F 1997 *Fractals and Fractional Calculus in Continuum Mechanics* ed A Carpinteri and F Mainardi (Berlin: Springer) pp 223–76
[25] Agrawal O 2006 A finite element scheme for fractional variational problems (submitted for review)